J. Fluid Mech. (1965), *vol.* 23, *part* 4, *pp.* 673–687
Printed in Great Britain

On similarity solutions of the boundary-layer equations with algebraic decay

By S. N. BROWN AND K. STEWARTSON
Department of Mathematics, University College, London, W.C. 1

(Received 14 April 1965)

Similarity solutions in which the vorticity decays algebraically towards the outer limit of the viscous layer are shown to be possible limit solutions of the full boundary-layer equations with exponential decay. The discussion is illustrated by consideration of a mainstream applicable to viscous flow in a cone.

1. Introduction

In a recent paper Ackerberg (1965) considers the steady, axisymmetric, converging motion of a viscous incompressible fluid inside an infinite circular cone. For large $r\nu/A$, where r is the distance from the vertex, A the volumetric flow rate and ν the kinematic viscosity, a solution is found by the Stokes method. For small $r\nu/A$ there is an inner expansion which satisfies the no-slip condition at the wall, and an outer expansion which satisfies a condition at the axis of the cone. Following Goldstein (1965) Ackerberg insists that these expansions match with an exponentially small error. This requirement implies that the outer solution near the apex is not potential sink flow, for which the outer velocity is $U \propto r^{-2}$. The smallest allowable singularity for which the inner solution has exponential decay has $U \propto r^{-3}$, and Ackerberg shows that the result of using this form of the outer velocity as a first approximation near the apex is a vortex motion with closed streamlines and a stagnation point on the axis. On physical grounds this situation does not appear likely to occur, and indeed it has not been observed experimentally.

In view of the fact that a potential sink flow seems a more acceptable solution from a practical standpoint and is considered to be applicable to flow in a cone by Jones and Watson (see Rosenhead 1963), a thorough investigation of the validity of the reasons for its rejection is necessary. In this paper we suggest that the solution of the associated similarity equation should not be disallowed merely because it has algebraic decay at the edge of the boundary layer. The similarity solution will be an asymptotic solution of the full boundary-layer equations and at best can be expected to be correct in the limit $r \to 0$. It is argued that algebraic decay although not permitted over a finite range of x, where x measures distance along one wall, may be allowable at singular points, and that similarity solutions with algebraic decay can be limit solutions of the full boundary-layer equations with exponential decay.

The investigations in support of the above argument are as follows. In §2 the behaviour of the solution of the full boundary-layer equations at large distances from the wall is discussed. This represents the limiting process $y \to \infty$. In §3 similarity solutions are considered as limiting solutions for some limit x_l of the variable x. If the double limit $x \to x_l$, $y \to \infty$ is the same taken in either order we have what we shall term a 'commutative'† approach to the limit resulting in exponential boundary-layer decay. However, if there is a non-commutative approach to the limit, algebraic decay of the similarity solution may result and the point $x = x_l$, $y \to \infty$ is a singular point. This argument is illustrated by consideration of a mainstream

$$U_1(x) = c(1-x)^m, \tag{1.1}$$

where $c > 0$, $m < 0$ are constants, and it is shown that if $m+1 < 0$ we have a commutative approach to the limit $x \to 1$, $y \to \infty$, but if $m+1 \geqslant 0$ the approach is non-commutative. That this non-commutative approach to the limit is not a sufficient ground for rejecting the similarity solution is supported by the investigations of §4. In this final section the limit $y \to \infty$, $x \to 1$ in this order is considered numerically by use of Görtler's (1955) series. Two examples are discussed. The first, with $U_1(x) \sim (1-x)^{-1}$ as $x \to 1$, is chosen to show that the method predicts the generally accepted limit behaviour as $x \to 1$, and because the singularity at $x = 1$ is more severe than in the case of flow in a cone for which, after applying the Mangler–Stepanov transformation, the appropriate mainstream velocity $U_1(x) \sim (1-x)^{-\frac{2}{3}}$ as $x \to 1$. It is this second example which is of chief interest, and again Görtler's series gives the same results for the skin friction and displacement thickness as $x \to 1$ as does the similarity solution, even though the latter has algebraic decay while the former is constructed from a sequence of functions with exponential decay. The evidence for permitting such algebraic decay at singular points thus seems fairly conclusive.

2. The boundary-layer equations and the solution at large distances from the wall

The equations describing the two-dimensional steady flow in the boundary layer of an incompressible fluid with a main stream $U_1(x)$ are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_1\frac{dU_1}{dx} + \frac{\partial^2 u}{\partial y^2}, \tag{2.2}$$

where x, y are measured along and perpendicular to the surface, respectively, and u, v are the corresponding velocity components. The variables v and y are 'scaled' variables and are related to the mainstream or outer variables v', y' by $v = v'/\nu^{\frac{1}{2}}$, $y = y'/\nu^{\frac{1}{2}}$, the symbol ν denoting the kinematic viscosity. The

† The word 'commutative' serves to distinguish between the type of double limit we have considered here, i.e. one in which the approach to the limiting point is along lines parallel to the co-ordinate axes only, and the more general double limit in which the approach to the limit may be made along any path in the (xy)-plane. If the structure of the solution is independent of the path in the second case we may define the double limit to be 'uniform'.

boundary conditions associated with these equations are that u is a given function $I(y)$ of y at some initial station of x which we can take to be $x = 0$, and

$$u = v = 0 \quad \text{at} \quad y = 0, \quad x > 0, \tag{2.3}$$

$$u \to U_1(x) \quad \text{as} \quad y \to \infty, \quad x > 0. \tag{2.4}$$

The manner in which $u \to U_1(x)$ as $y \to \infty$ is discussed by Goldstein (1965). He notes that to find higher approximations to the Navier–Stokes equations than are implied by (2.1), (2.2) the correct procedure is to match an expansion for small viscosity in the outer variables x, y' to one in the inner or scaled variables x, y. Expansions for the stream function ψ', defined by $u = \partial\psi'/\partial y'$, $v' = -\partial\psi'/\partial x$, in powers of $\nu^{\frac{1}{2}}$ might first be tried with

$$\left.\begin{aligned} \psi'_{\text{outside}} &= g_0(x, y') + \nu^{\frac{1}{2}} g_1(x, y') + \nu g_2(x, y') + \ldots, \\ \psi'_{\text{inside}} &= \nu^{\frac{1}{2}}[f_0(x,y) + \nu^{\frac{1}{2}} f_1(x,y) + \nu f_2(x,y) + \ldots]. \end{aligned}\right\} \tag{2.5}$$

As Goldstein says: 'It is known that expansions of these forms are not sufficiently general, but the simplest possible case will serve to illustrate the matter under discussion.' In (2.5) the asymptotic expansions of $f_0, f_1, f_2, \ldots$ for large y provide the values of $g_1, g_2, g_3, \ldots$ as $y' \to 0$. If a term y^{-N} occurs in the asymptotic expansion of f_0 then the term in $\nu^{\frac{1}{2}N}$ in ψ'_{outside} would behave like y'^{-N} as $y' \to 0$. Since all subsequent g_i are harmonic if g_0 is harmonic, and it is impossible for a harmonic function to be infinite along a finite portion of the x-axis, Goldstein excludes such algebraic decay and rewrites the condition (2.4) as

$$y^N\left(\frac{u}{U_1} - 1\right) \to 0 \quad \text{as} \quad y \to \infty \tag{2.6}$$

for all real N. However, the investigations of this paper seem to indicate that non-exponential decay is permissible at singular points in the event of a non-commutative approach to the double limit $x \to a$, $y \to \infty$, where a is either 0, 1 or ∞. It is convenient at this stage to consider the limit $y \to \infty$ and discuss the form taken by the solution at the edge of the boundary layer.

If we consider a boundary layer that at some stage, say $x = 0$, had either a Blasius- or stagnation-type velocity profile, it is to be expected that for positive x and large y the velocity components may be written

$$u = U_1(x) + A(x,y)\exp\left[-\frac{(y-k(x))^2}{2F(x)}\right] + \ldots, \quad v = -yU_1'(x) + h'(x) + \ldots. \tag{2.7}$$

Here k, F, h' are functions of x alone and A is algebraic in y, i.e. it can be written as a series of descending powers of y whose coefficients are functions of x. Further the terms neglected in (2.7) are exponentially smaller than those retained. If (2.7) is substituted into (2.2) the terms of highest order lead to

$$AU_1' + U_1\left[\frac{\partial A}{\partial x} + \frac{AF'}{2F^2}(y-k)^2 + \frac{Ak'}{F}(y-k)\right] + (h' - yU_1')\left[\frac{\partial A}{\partial y} - \frac{A}{F}(y-k)\right]$$
$$= \frac{\partial^2 A}{\partial y^2} - \frac{2}{F}(y-k)\frac{\partial A}{\partial y} + A\left[\frac{(y-k)^2}{F^2} - \frac{1}{F}\right]. \tag{2.8}$$

When the coefficients of y^2, y and 1 in this expression are equated to zero the following equations result for the functions $F(x)$, $k(x)$, $A(x,y)$:

$$\tfrac{1}{2}U_1F' + U_1'F = 1, \tag{2.9}$$

$$U_1k' + U_1'k = h', \tag{2.10}$$

$$U_1\frac{\partial A}{\partial x} + \left(\frac{2}{F} - U_1'\right) y\frac{\partial A}{\partial y} + A\left(U_1' + \frac{1}{F}\right) = 0. \tag{2.11}$$

From (2.9), (2.10) we obtain immediately

$$\left.\begin{aligned} F(x) &= \frac{2\int_{x_1}^{x} U_1(x)\,dx}{U_1^2}, \\ k(x) &= (h(x)+k_1)/U_1(x), \end{aligned}\right\} \tag{2.12}$$

x_1, k_1 being constants of integration, while the solution of (2.11) is

$$A = G(x)\,K(yL(x)),$$

where K is an arbitrary function. This leads to

$$G(x) = \frac{\alpha_1}{U_1\left(\int_{x_1}^{x} U_1\,dx\right)^{\frac{1}{2}}}, \quad L(x) = \frac{\alpha_2 U_1}{\int_{x_1}^{x} U_1\,dx}, \tag{2.13}$$

where α_1, α_2 are constants. Since A is an algebraic function of y its form when y is large is

$$A = \frac{\alpha}{U_1\left(\int_{x_1}^{x} U_1\,dx\right)^{\frac{1}{2}}}\left(\frac{U_1 y}{\int_{x_1}^{x} U_1\,dx}\right)^n, \tag{2.14}$$

where n is a constant. However, the governing equations (2.1), (2.2) are parabolic, so an essential additional condition is a prescribed velocity profile at an initial station of x. This will determine the values of the exponent n in (2.14) and the constant x_1. If the initial profile is that of Blasius, for example, so that as $x \to 0$ for large y

$$u \approx U_0 - \frac{2\gamma_1(U_0x)^{\frac{1}{2}}}{y}\exp\left[-\frac{1}{4}\left\{y - \beta_1\left(\frac{x}{U_0}\right)^{\frac{1}{2}}\right\}^2\frac{U_0}{x}\right], \tag{2.15}$$

(see Schlichting 1960), where $U_0 = U_1(0)$, $\beta_1 = 1{\cdot}72$, $\gamma_1 = 0{\cdot}231$, it follows that $n = -1$, $x_1 = 0$. On the other hand if we start with a stagnation-point profile so that $U_1(x) = U_0x$ for small x, then the initial profile for large y takes the form

$$u \approx U_0x - \frac{cx}{y^3}\exp\left[-\frac{1}{2}\left(y - \frac{\delta_1}{\sqrt{U_0}}\right)^2 U_0\right], \tag{2.16}$$

where $\delta_1 = 0{\cdot}648$. Thus $n = -3$, $x_1 = 0$ in this case. Similar considerations of the form of the flow at the outer edge of the boundary layer have been made by Tollmien (see Rosenhead 1963), but the co-ordinates employed are not suited to our purpose. The function $k(x)$ may readily be identified with the displacement thickness $\delta^*(x)$ defined by

$$\delta^*(x) = \int_0^\infty \left(1 - \frac{u}{U_1(x)}\right)dy. \tag{2.17}$$

From (2.7) we see that for large y the stream function

$$\psi \approx U_1(x)\,y - h(x) - \text{const.}, \tag{2.18}$$

so that

$$\delta^*(x) = \left[y - \frac{\psi}{U_1(x)}\right]_0^\infty = \frac{h(x) + \text{const.}}{U_1(x)}. \tag{2.19}$$

Since for the two profiles given by (2.15), (2.16) we know that $\beta_1(x/U_0)^{\frac{1}{2}}$, $\delta_1/U_0^{\frac{1}{2}}$ are respectively the displacement thicknesses, we deduce that for any boundary layer that initially had a Blasius or stagnation-point type profile the constant in (2.19) is equal to k_1 in (2.12).

If instead of (2.7) the exponentially decaying velocity profile is taken as

$$u = U_1(x) + A(x,y)\exp\left[-\frac{(y-k(x))^\alpha}{2F(x)}\right] + \ldots, \quad v = -yU_1'(x) + h'(x) + \ldots \quad (\alpha > 0), \tag{2.20}$$

it can be shown that the only other possible value of α is 1. When the expressions (2.20) are substituted into (2.2) we obtain

$$AU_1' + U_1\left[\frac{\partial A}{\partial x} + \frac{AF'}{2F^2}(y-k)^\alpha + \frac{A\alpha k'}{2F}(y-k)^{\alpha-1}\right] + (h' - yU_1')\left[\frac{\partial A}{\partial y} - \alpha\frac{(y-k)^{\alpha-1}}{2F}A\right]$$
$$= \frac{\partial^2 A}{\partial y^2} - \frac{\alpha}{F}(y-k)^{\alpha-1}\frac{\partial A}{\partial y} + A\left[\frac{\alpha^2}{4F^2}(y-k)^{2\alpha-2} - \frac{\alpha(\alpha-1)(y-k)^{\alpha-2}}{2F}\right]. \tag{2.21}$$

It follows, on equating to zero the coefficient of y^α, that

$$\begin{aligned}(U_1F'A/2F^2) + (\alpha U_1'A/2F) &= 0 \quad \text{if} \quad \alpha < 2\\ &= A\alpha^2/4F^2 \quad \text{if} \quad \alpha = 2,\end{aligned} \tag{2.22}$$

while $\alpha > 2$ is incompatible with the original assumption (2.8). The case $\alpha = 2$ has been dealt with already, and if $\alpha < 2$ (2.22) gives

$$F \propto U_1^{-\alpha}. \tag{2.23}$$

Consideration of the terms of the next highest order of magnitude shows that $\alpha = 1$, in which case it is simpler to dispense with the function $k(x)$ and write

$$u = U_1(x) + A(x,y)\,e^{-y/F(x)} + \ldots, \quad v = -yU_1'(x) + h'(x) + \ldots. \tag{2.24}$$

The coefficient of y^0 in (2.21) with $\alpha = 1$, $k = k' = 0$ gives

$$AU_1' + U_1\frac{\partial A}{\partial x} - U_1'y\frac{\partial A}{\partial y} - \frac{A}{2}\left(\frac{h'}{F} + \frac{1}{2F^2}\right) = 0, \tag{2.25}$$

to which, if required, the solution in the form $A = G(x)\,K\{yL(x)\}$ with $L \propto U_1(x)$ may be found. The function $G(x)$ involves the function $h'(x)$ which can only be determined when the asymptotic form (2.24) is matched to the solution for small y. An example of exponential decay of the form (2.24) is given by the flow in a convergent channel with vertex at $x = 1$ for which $U_1(x) = 1/(1-x)$ and (see Schlichting 1960)

$$\begin{aligned}u/U_1 &= 3\tanh^2\{(y/(1-x)\sqrt{2}) + 1{\cdot}146\} - 2\\ &\approx 1 - 1{\cdot}212\,e^{-y\sqrt{2}/(1-x)} \quad \text{as} \quad y \to \infty.\end{aligned} \tag{2.26}$$

3. The role of the similarity solution in boundary-layer theory

'Similar' solutions of equations (2.1), (2.2) are defined as those for which two velocity profiles $u(x, y)$ located at different stations of x differ only by scale factors multiplying u and y. The scale factor for u is $U_1(x)$, the associated main-stream velocity, and then the above definition requires that

$$u/U_1(x) = \text{a function of } \eta, \text{ where } \eta = yg(x), \tag{3.1}$$

for some function $g(x)$. Similarity solutions of (2.1), (2.2) are possible, as noted by Goldstein (1939, 1965), if $U_1 = c\,e^{kx}$ or $c\,e^{-kx}$, or if

$$U_1(x) = c(l \pm x)^m, \tag{3.2}$$

where c, l, m, k are constants, as long as $U_1(x)$ given by (3.2) is real.

The great advantage of similarity solutions as defined above is the reduction of the partial differential equations of the problem to one ordinary third-order differential equation which, although still non-linear, is a considerable mathematical simplification. However, the limitations of such solutions must be realized. The partial differential equations do not impart their parabolic nature to the ordinary differential equation on whose solution it is not possible to impose an arbitrary condition at an initial station of x, since all values of x are now equivalent. The conclusion is then, that these similarity solutions are in a sense asymptotic, and will only be correct in some limit which may never be attained. In any particular application of a similarity solution this limitation of the solution should be realized, and the limit in which it is asymptotically correct should be carefully considered.

We now examine possible limiting forms of $U_1(x)$, and discuss the regions either finite or infinitesimal, in which we may expect the associated similarity solution to give an asymptotically correct picture of the flow. The most familiar of all similarity solutions is probably that of Blasius for a flat plate in which $U_1(x) = U_0$. There is no loss of generality in taking $U_0 = 1$, and we note that this could be a limiting form of $U_1(x)$ as x tends either to zero, infinity, or a finite intermediate value. Where then will the similarity solution in terms of the variable $\eta = y/\sqrt{x}$ be a relevant solution of the boundary-layer equations? The answer to this question depends not only on $U_1(x)$ but also on $I(y)$, the value of u at $x = 0$. If $I(y) \equiv 1$ for all $y > 0$ it is relevant as far as the first value of x at which U_1 differs from unity. If $I(y) \not\equiv 1$ for all $y > 0$ then no matter how U_1 varies it cannot be the relevant solution for any finite x, for by anology with the heat equation the effect of the non-uniformity in I takes an infinite distance to disappear. At best, therefore, it will be relevant only in the limit $x \to \infty$, and even then only if $U_1(\infty) = 1$. The velocity profile for large y is given by (2.15), and attention is drawn to the fact that the exponential decay $\exp(-\frac{1}{4}y^2/x)$ is as given by (2.7) and (2.12) with $F(x) = 2x$. Thus the similarity solution can, and does, give the right x-dependence of the exponential decay as forecast by (2.7). If we assume for the sake of argument that the Blasius solution holds only as $x \to \infty$ we have a commutative approach to the double limit $x \to \infty$, $y \to \infty$. The limit $x \to \infty$ followed by $y \to \infty$ is given by the similarity solution and (2.15), while the limit $y \to \infty$ followed by $x \to \infty$ is given by (2.7) and (2.12). In this case the limits are the same. If we appear to labour this point it is because the concept of these double limits is essential to our subsequent arguments.

Similarity solutions, with $U_1 = cx^m$ $(c > 0)$, are characterized by

$$u = U_1 F_1'(\eta), \quad g(x) = \left(\frac{U_1}{x}\right)^{\frac{1}{2}} = c^{\frac{1}{2}} x^{\frac{1}{2}(m-1)}, \tag{3.3}$$

which lead to the familiar Falkner–Skan (1930) equation

$$F_1''' + \tfrac{1}{2}(m+1)\, F_1 F_1'' - mF_1'^2 + m = 0. \tag{3.4}$$

The solution of (3.4) with boundary conditions $F_1(0) = F_1'(0) = 0$, $F_1'(\infty) = 1$ will again be a limit solution of the boundary-layer equations. If $m > 0$ we can expect the solution to hold in the limit $x \to 0$ if $U_1(x)/x^m \to c$ as $x \to 0$ and $I(y) \equiv 0$. It will be a limit solution as $x \to \infty$ if $U_1 \approx cx^m$ for large x and $I(y) \not\equiv 0$. If $m < 0$, (3.3), (3.4) constitute an asymptotic solution in the limit $x \to \infty$ and then only if separation has not previously occurred. The similarity solution predicts separation when $m = -0{\cdot}0904$. For $m+1 > 0$ an appropriate change of variables is

$$Y = [\tfrac{1}{2}(m+1)]^{\frac{1}{2}}\, \eta, \quad f(Y) = [\tfrac{1}{2}(m+1)]^{\frac{1}{2}}\, F_1(\eta), \tag{3.5}$$

and then equation (3.4) becomes

$$f''' + ff'' + \beta(1 - f'^2) = 0, \tag{3.6}$$

where

$$\beta = 2m/(m+1). \tag{3.7}$$

The solutions of (3.6) for $m \geqslant -0{\cdot}0904$ have been tabulated by Hartree (1937). For large Y the solution takes the form

$$u/U_1(x) \sim 1 + A_1 e^{-\frac{1}{2}\zeta^2} \zeta^{-2\beta-1} + B_1 \zeta^{2\beta}, \tag{3.8}$$

where $\zeta = Y + b$ and A_1, B_1, b are constants. If $\beta > 0$, B_1 must be zero and the exponential decay $e^{-\frac{1}{2}\zeta^2} \sim \exp[-\tfrac{1}{4}c(m+1)\, x^{m-1} y^2]$ given by (3.8) is as predicted by (2.7) with

$$F(x) = 2\int_0^x U_1\, dx \Big/ U_1^2 = \frac{2x^{1-m}}{c(m+1)}. \tag{3.9}$$

Thus as in the case of the Blasius profile the sequence of limits $x \to \infty$, $y \to \infty$ and $y \to \infty$, $x \to \infty$ give the same result. However, if $\beta < 0$ there is now no need for B_1 to be zero in order to satisfy the condition $u \to U_1(x)$ as $y \to \infty$ and the solutions are not necessarily unique. If the extra condition of exponential decay is added, two solutions are obtained with the correct exponential behaviour at infinity as predicted by the full boundary-layer equations. The one found by Stewartson (1954) has $f''(0) < 0$, and if on physical grounds back-flow is to be excluded, we infer, though cannot prove, that the other is the required solution. Thus in this case we have chosen the solution so that the limit $x \to \infty$, $y \to \infty$ is the same taken in either order.

In addition to the above limit solutions there are those for which x tends to a finite non-zero value. These are given by

$$U_1(x) = c(1-x)^m \quad (c > 0), \tag{3.10}$$

where, without loss of generality, we have taken $l = 1$ in (3.2). The associated similarity solution we expect to be asymptotically correct in the limit $x \to 1$. The appropriate form is

$$u = U_1(x)\, F_1'(\eta), \quad g(x) = c^{\frac{1}{2}}(1-x)^{\frac{1}{2}(m-1)}, \tag{3.11}$$

where

$$F_1''' - \tfrac{1}{2}(m+1)\, F_1 F_1'' + mF_1'^2 - m = 0. \tag{3.12}$$

The term 'backward boundary layer' is used by Goldstein to denote a boundary layer which has been influenced by viscosity through an infinite distance. Such terminology will not be employed here; instead the point $x = 1$ when $U_1(x) = c(1-x)^m$ will be regarded as the 'end-point' of an ordinary forward boundary layer. Of course if $m > 0$ we do not expect the fluid to attain this end-point as separation will have taken place for some $x < 1$. There are in fact no solutions of (3.12) for $m > 0$ which satisfy

$$F_1(0) = F_1'(0) = 0, \quad F_1'(\infty) = 1. \tag{3.13}$$

If $m+1 > 0$ we make the transformation (3.5) and equation (3.12) becomes

$$f''' - ff'' - \beta(1-f'^2) = 0, \tag{3.14}$$

where again $\beta = 2m/(m+1)$. If, however, $m+1 < 0$ we write

$$Y = [-\tfrac{1}{2}(m+1)]^{\frac{1}{2}}\eta, \quad f(Y) = [-\tfrac{1}{2}(m+1)]^{\frac{1}{2}} F_1(\eta) \tag{3.15}$$

and obtain equation (3.6). The solution for large Y of equation (3.6) is given by (3.8), while in the case $m+1 > 0$ the asymptotic form is, from equation (3.14),

$$u/U_1(x) \sim 1 + A_1 e^{\frac{1}{2}\zeta^2}\zeta^{-2\beta-1} + B_1\zeta^{2\beta} \quad \text{(see Goldstein 1965).} \tag{3.16}$$

The condition at infinity can only be satisfied if $A_1 = 0$ and $\beta < 0$ $(m < 0)$ in which case the decay is algebraic. Goldstein excludes such algebraic decay but the investigations of the following section give support to our conjecture that similarity solutions with algebraic decay can be limit solutions of the full boundary equations with exponential decay.

The solution for large y of the full boundary-layer equations is given by (2.7) with

$$F(x) = \frac{2\int_0^x (1-x)^m\, dx}{c(1-x)^{2m}} = \frac{2(1-x)^{-2m}}{c(m+1)}[1-(1-x)^{m+1}], \tag{3.17}$$

and if $m+1 < 0$, as $x \to 1$,

$$F(x) \sim \{-2/c(m+1)\}\,(1-x)^{1-m}, \tag{3.18}$$

but if $m+1 > 0$, as $x \to 1$,

$$F(x) \sim \{2(1-x)^{-2m}/c(m+1)\}. \tag{3.19}$$

The convergence or otherwise of the integral in (3.7) as $x \to 1$ is seen to be crucial. Suppose first that $m+1 < 0$. Then the form of $F(x)$ as $x \to 1$, given by (3.18), implies that the exponent of the exponential decay of $U_1 - u$ in (2.7) is

$$\tfrac{1}{4}(m+1)\, cy^2(1-x)^{m-1}, \tag{3.20}$$

which contains the similarity variable $yg(x)$, where $g(x)$ is given by (3.11). Thus the double limit $x \to 1$, $y \to \infty$ is commutative.

Suppose now that $m+1 > 0$. Then the form of $F(x)$ as $x \to 1$, given by (3.19), implies that the exponential decay of $U_1 - u$ in (2.7) is

$$-\tfrac{1}{4}(m+1)\,cy^2(1-x)^{2m}, \tag{3.21}$$

which does *not* contain the similarity variable $yg(x)$. Thus if $1 > m+1 > 0$ the similarity solution cannot possibly give the correct exponential behaviour as $y \to \infty$, quite apart from its detailed properties. Thus the double limit is non-commutative, but there is no reason to suppose that this, of itself, invalidates the results given by the similarity solution for finite η. Indeed in the Appendix an illustrative differential equation is solved which points the way to the reconciliation of the different decay properties in the two limiting processes. The discussion there leads one to expect that as $x \to 1$, with $y(1-x)^{\frac{1}{2}(m-1)}$ fixed, the similarity solution is correct and $u - U_1$ algebraically small if

$$(1-x)^{-m} \gg y \gg (1-x)^{-\frac{1}{2}(m-1)}. \tag{3.22}$$

As $x \to 1$, with $y(1-x)^m$ fixed, the similarity solution is not correct; $u - U_1$ is algebraically small if $y(1-x)^m$ is small and (3.22) holds, but is exponentially small if $y(1-x)^m$ is large.

The question might now be asked that if algebraic decay is permitted in this case why is it not permitted in the solutions of Hartree for $U_1(x) = cx^m$ with $0 > m > -0{\cdot}0904$? A definite answer to this question cannot be given, although the difference between the two situations should be noted. When $U_1(x) = cx^m$ the exponential decay solution is selected because there is a choice and it seems reasonable to choose the solution with the same exponential decay as the solution of the full boundary-layer equations.† However, when

$$U_1(x) = c(1-x)^m \quad (m+1 > 0),$$

there is no choice of solution; the algebraic one is unique and the similarity solution could not in any case give the correct exponential decay.

Another example of a non-commutative approach to the double limit $x \to 1$, $y \to \infty$ may be noted. Suppose that $U_1(x) = 1/(1-x)$ and that at $x = 0$ a Blasius velocity-profile is prescribed. Then the exponential decay will be of the e^{-y^2} form given initially by (2.7). However, the similarity solution (2.26) gives an e^{-y} exponential decay. This non-commutative approach has not been regarded as exceptional because in both cases the decay is exponential.

4. Solution by Görtler's series

The numerical evidence in support of the assertion that similarity solutions with algebraic decay need not necessarily be disallowed is now presented. In view of the fact that the subject of viscous flow in a cone is of immediate interest we consider an example in which $U_1(x) \sim (1-x)^{-\frac{2}{3}}$ as $x \to 1$, where $x = 1$ is the vertex of the cone. The exponent $-\frac{2}{3}$ is obtained on using Mangler's trans-

† Numerical evidence to support the choice of the exponentially decaying solution is now available and will be published elsewhere.

formation which reduces the axisymmetric boundary-layer equations with a mainstream $\bar{U}_1(\bar{x}) = (1-\bar{x})^{-2}$ to two-dimensional form with a mainstream $U_1(x) = (1-x)^{-\frac{2}{3}}$. The method of procedure is to compute the skin friction and displacement thickness by Görtler's series, let $x \to 1$ in the final results and compare them with the values obtained by the solution of the associated similarity equation. It is of interest to note that the independent variable in the equations used by Görtler to compute the terms of the series is

$$\bar{\eta} = yU_1(x)\bigg/\left\{2\int_0^x U_1(x)\,dx\right\}^{\frac{1}{2}}, \tag{4.1}$$

which is the variable $y/\sqrt{\{F(x)\}}$ obtained in (2.7), (2.12). Thus indeed Görtler's series will allow us to take the limit $y \to \infty$, $x \to 1$ in that order, where the limit $y \to \infty$ is as defined by the investigation of § 2. The similarity solution represents the order $x \to 1$ followed by $y \to \infty$.

In order to test the accuracy of this application of Görtler's series we first consider an example with a more severe singularity as $x \to 1$. This is the two-dimensional flow in a convergent channel for which the associated similarity solution is generally accepted to give correct results although, as pointed out earlier, there is again a non-commutative approach to the double limit though in both cases the decay is exponential. Görtler noted that because of the singularity in the solution at separation when $x \simeq 0{\cdot}16$ for $U_1(x) = 1/(1+x)$, the series for $U_1(x) = 1/(1-x)$ also has a limited radius of convergence because, although there can be no separation, the associated series will have a singularity on the negative real axis due to its similarity to that for $U_1(x) = 1/(1+x)$. Thus it is essential to choose $U_1(x)$ so that, not only does it have the right form as $x \to 1$, but leads to a convergent series. By trial it is found that convenient forms to consider are

$$U_1(x) = \frac{x}{1-x^2} \quad \text{and} \quad U_1(x) = \frac{x}{(1-x^2)^{\frac{2}{3}}}. \tag{4.2}$$

The stagnation point at $x = 0$ is not a disadvantage, and possible physical situations which these flows could represent are illustrated in figures 1(*a*), (*b*), where figure 1(*a*) is probably more appropriate for the two-dimensional flow, while figure 1(*b*) could represent one of a series of cones in an infinite reservoir. In each case we consider the boundary layer on the shaded wall from $x = 0$ to $x = 1$.

(*a*) $U_1(x) = x/(1-x^2)$.

The similarity equation for the convergent channel flow is (3.12) with $m = -1$ and $\eta = 2^{-\frac{1}{2}}(1-x)^{-1}y$, the factor $2^{-\frac{1}{2}}$ following from the fact that $U_1(x) \sim 1/2(1-x)$ as $x \to 1$. The solution is given by Schlichting (1960), and the formulae for the skin friction and displacement thickness are

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 2^{-\frac{3}{2}}(1-x)^{-2}F_1''(0) = 6^{-\frac{1}{2}}(1-x)^{-2} \tag{4.3}$$

since $F_1''(0) = \frac{2}{3}$, and

$$\delta^* = \int_0^\infty \left(1-\frac{u}{U}\right)dy = 2^{\frac{1}{2}}(1-x)\int_0^\infty (1-F_1')\,d\eta = 2(3-6^{\frac{1}{2}})(1-x). \tag{4.4}$$

When the skin friction is calculated from Görtler's series we obtain

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = \frac{x^2}{(1-x^2)^2\sqrt{(2\xi)}}(1\cdot23259+1\cdot48152\xi-0\cdot54000\xi^2+0\cdot46857\xi^3 -0\cdot510111\xi^4+0\cdot62856\xi^5), \quad (4.5)$$

where ξ is the auxiliary variable given by

$$\xi = \int_0^x U_1(x)\,dx = -\tfrac{1}{2}\log(1-x^2). \quad (4.6)$$

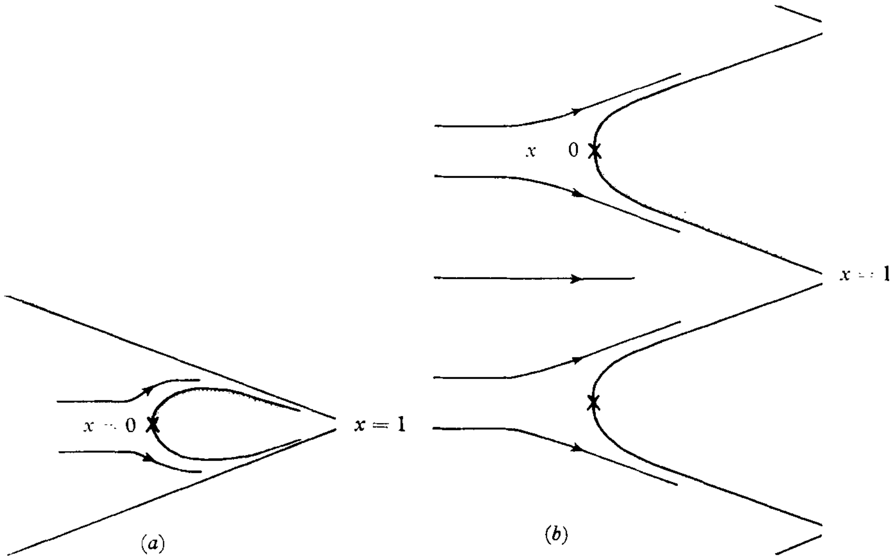

FIGURE 1 (*a*), (*b*). Possible physical situations appropriate to the mainstreams given in equation (4.2).

Since $x \to 1$ corresponds to $\xi \to \infty$, it is advisable to investigate the convergence of (4.5) in terms of the original variable x. The expression becomes

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = \frac{x}{(1-x^2)^2}(1\cdot23259+0\cdot43261x^2-0\cdot03969x^4+0\cdot00948x^6 -0\cdot00288x^8+0\cdot00111x^{10}-\ldots). \quad (4.7)$$

The coefficients of this series are decreasing and the terms are of opposite sign, so we may expect it to converge even at $x = 1$. The last three partial sums are

$$1\cdot6350, \quad 1\cdot6321, \quad 1\cdot6333 \quad (4.8)$$

so we deduce that, as $x \to 1$,

$$(1-x)^2\left.\frac{\partial u}{\partial y}\right|_{y=0} \to \frac{1\cdot633}{4} = 0\cdot408, \quad (4.9)$$

which agrees with (4.3) to three places of decimals.

The displacement thickness δ^* is obtained as the series in ξ

$$\delta^* = \frac{(1-x^2)}{x\sqrt{(2\xi)}}(0\cdot64790-0\cdot55174\xi+0\cdot56986\xi^2-0\cdot65318\xi^3 +0\cdot82695\xi^4-1\cdot11888\xi^5), \quad (4.10)$$

and when put in terms of the original variable x this becomes

$$\delta^* = (1-x^2)(0{\cdot}64790 - 0{\cdot}11390x^2 + 0{\cdot}02330x^4 - 0{\cdot}00831x^6 + 0{\cdot}00261x^8 - 0{\cdot}00083x^{10}). \quad (4.11)$$

Again the coefficients are decreasing and of opposite sign. On letting $x \to 1$ we deduce that the sum of the series lies between $0{\cdot}5516$ and $0{\cdot}5508$. Thus, we estimate

$$(1-x)^{-1}\,\delta^* \to 1{\cdot}102 \quad \text{as} \quad x \to 1, \quad (4.12)$$

while (4.4) gives

$$(1-x)^{-1}\,\delta^*_{x=1} = 1{\cdot}101, \quad (4.13)$$

and (4.12) differs from this by about $0{\cdot}1$ %.

The above seems a convincing test for the application of Görtler's series to such a situation. There is no point in verifying the result for the momentum thickness, as this can be expressed in terms of the two quantities already found.

(*b*) $U_1(x) = x/(1-x^2)^{\frac{2}{3}}$.

The similarity equation for this mainstream velocity as $x \to 1$ is (3.12) with $m = -\frac{2}{3}$, or on making the change of variable (3.5),

$$f''' - ff'' + 4(1-f'^2) = 0, \quad (4.14)$$

which is (3.14) with $\beta = -4$. The asymptotic form of the solution is given by (3.16) and the required solution is that for which $A_1 = 0$. A table of f' is given in Rosenhead (1963). The value of $f''(0)$ is $2{\cdot}273$ and $\int_0^\infty (1-f')\,dY$ is equal to $0{\cdot}410$. Since the independent variable Y of equation (4.14) is given in terms of the original variable y by

$$y = 2^{\frac{1}{3}}(1-x)^{\frac{5}{6}}\,\eta = 6^{\frac{1}{2}}2^{\frac{1}{3}}(1-x)^{\frac{5}{6}}\,Y \quad (4.15)$$

and

$$u = 2^{-\frac{2}{3}}(1-x)^{-\frac{2}{3}}f'(Y),$$

we have

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = \frac{1}{2\sqrt{6}}(1-x)^{-\frac{3}{2}} \times 2{\cdot}273. \quad (4.16)$$

Also

$$\delta^* = 2^{\frac{5}{6}}\sqrt{3}(1-x)^{\frac{5}{6}} \times 0{\cdot}410. \quad (4.17)$$

The expression for the skin friction calculated from Görtler's series is

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = \frac{x^2}{(1-x^2)^{\frac{4}{3}}\sqrt{(2\xi)}}(1{\cdot}23259 + 1{\cdot}48152\lambda + 1{\cdot}00847\lambda^2 + 0{\cdot}87174\lambda^3 + 0{\cdot}69875\lambda^4 + 0{\cdot}60958\lambda^5), \quad (4.18)$$

where

$$\tfrac{3}{2}\lambda = \xi = \int_0^x U_1(x)\,dx = \tfrac{3}{2}[1-(1-x^2)^{\frac{1}{3}}]. \quad (4.19)$$

However, we suspect that the singularity in $\partial u/\partial y|_{y=0}$ as $x \to 1$ is $(1-x^2)^{-\frac{3}{2}}$. With this as a factor (4.18) becomes, in terms of x,

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = \frac{x}{(1-x^2)^{\frac{3}{2}}}(1{\cdot}23259 + 0{\cdot}08298x^2 - 0{\cdot}00208x^4 - 0{\cdot}00039x^6 - 0{\cdot}00026x^8 - 0{\cdot}00016x^{10}). \quad (4.20)$$

Even as $x \to 1$ the terms of this expression are rapidly decreasing. The last three partial sums are

$$1{\cdot}3135, \quad 1{\cdot}3131, \quad 1{\cdot}3129, \quad (4.21)$$

and fitting a polynomial in $1/n$ to these we estimate that

$$(1-x)^{\frac{3}{2}}\, \partial u/\partial y\big|_{y=0} \to 2^{-\frac{3}{2}} \times 1{\cdot}312 = 2{\cdot}272/2\sqrt{6}. \tag{4.22}$$

This differs from (4.16) by less than 0·1 %.

For the displacement thickness, Görtler's series gives

$$\delta^* = \frac{\sqrt{(2\xi)}\,(1-x^2)^{\frac{4}{3}}}{x}(0{\cdot}64790 + 0{\cdot}55174\lambda - 0{\cdot}04175\lambda^2 + 0{\cdot}05576\lambda^3 - 0{\cdot}02397\lambda^4 + 0{\cdot}01681\lambda^5), \tag{4.23}$$

which may be written

$$\delta^* = (1-x^2)^{\frac{5}{6}}(0{\cdot}64790 + 0{\cdot}03205x^2 + 0{\cdot}01401x^4 + 0{\cdot}00360x^6 + 0{\cdot}00419x^8 + 0{\cdot}00258x^{10}). \tag{4.24}$$

These terms are not decreasing as rapidly as those of (4.20) but it is expected that the series is convergent even as $x \to 1$. The last three partial sums are

$$0{\cdot}6976, \quad 0{\cdot}7018, \quad 0{\cdot}7044, \tag{4.25}$$

and a polynomial in $1/n$ fitted to these three gives 0·7114, so that as $x \to 1$

$$(1-x)^{-\frac{5}{6}}\,\delta^* \to 2^{\frac{5}{6}}\sqrt{3} \times 0{\cdot}411, \tag{4.26}$$

which again differs from (4.17) by only 1 in the third place. Thus the similarity solution and the expansion method agree to a high degree of accuracy. It seems then that a similarity solution with algebraic decay can be the limit of a solution of the full boundary-layer equation, for in this one case at least the different limiting processes lead to the same result. The only singular point is $x = 1$, $y \to \infty$, where the approach to the double limit is non-uniform. Further, since $u \to U_1$ through exponentially small terms as $y \to \infty$ except at one point only, the difficulty envisaged by Goldstein in matching the boundary layer with the inviscid flow outside will not be encountered.

Appendix

An illustrative differential equation

The following investigation of the analytic form of the non-commutative limit discussed in §§ 2, 3 was made on a suggestion by Dr N. C. Freeman. Although the boundary-layer equations appropriate to flow in a cone are linearized, the solution obtained exhibits the predicted behaviour of exponential decay at the edge of the boundary layer except at the one point $x = 1$, corresponding to the vertex, where the approach to the limit is non-uniform.

We write

$$u = U_1 - u', \quad v = -yU_1'(x) + h'(x) + v' \tag{A 1}$$

in equation (2.2), assume that u', v' are small, and obtain the linearized equation

$$u'\frac{dU_1}{dx} + U_1\frac{\partial u'}{\partial x} + (h'(x) - yU_1')\frac{\partial u'}{\partial y} = \frac{\partial^2 u'}{\partial y^2}. \tag{A 2}$$

In order to satisfy as far as possible the requirements of boundary-layer theory, the boundary conditions for equation (A 2) will be taken as

$$\left.\begin{aligned} u' &= 0, \quad x = 0, \quad y > 0,\\ u' &\to 0, \quad y \to \infty, \quad x > 0,\\ u' &= U_1, \quad y = 0, \quad x > 0,\end{aligned}\right\} \tag{A 3}$$

although it is realized that the last condition is incompatible with the previous assumption that u' is small. However, it will be seen that the equation (A 2) with the boundary conditions (A 3), which imply that u is zero on the boundary, attains its mainstream value at the edge of the boundary layer, and that the initial profile is that of Blasius, is a suitable representation for illustrating the non-uniform approach to the limit.

If in (A 2) we substitute

$$U_1 = (1-x)^{-\frac{2}{3}}, \quad u' = (1-x)^{-\frac{2}{3}} H(x, \eta), \quad h'(x) = B(1-x)^{-\frac{5}{6}}, \tag{A 4}$$

where $\eta = y(1-x)^{-\frac{5}{6}}$, and B is a constant, the equation becomes

$$\frac{\partial^2 H}{\partial \eta^2} = (\tfrac{1}{6}\eta + B)\frac{\partial H}{\partial \eta} + (1-x)\frac{\partial H}{\partial x} + \tfrac{4}{3}H, \tag{A 5}$$

with boundary conditions

$$\left.\begin{aligned} H &= 0, \quad x = 0, \quad \eta > 0, \\ H &\to 0, \quad \eta \to \infty, \quad x > 0, \\ H &= 1, \quad \eta = 0, \quad x > 0. \end{aligned}\right\} \tag{A 6}$$

If the term $(1-x)\,\partial H/\partial x$ in equation (A 5) is neglected as $x \to 1$, for large values of η, the form of the solution is as given by the similarity equation (3.14) with $\beta = -4$. As $\eta \to \infty$, $H = O(\eta^{-8})$, and the decay is algebraic. However, the investigations of §2 indicate that equation (A 2) leads to a solution in which u' tends to zero exponentially at the edge of the boundary layer. To reconcile these two statements it is necessary to consider the complete solution of (A 5) with boundary conditions (A 6) which may readily be shown to be, with B set equal to zero,

$$H = \frac{1}{\sqrt{\pi}} \int_\theta^\infty \frac{e^{-\mu}\mu^{\frac{7}{2}}\,d\mu}{\left(\dfrac{y^2}{12(1-x)^{\frac{5}{3}}} + \mu\right)^4}, \tag{A 7}$$

where
$$\theta = \frac{y^2}{12[(1-x)^{\frac{4}{3}} - (1-x)^{\frac{5}{3}}]}. \tag{A 8}$$

When $1 - x \ll 1$ the behaviour of H for different régimes of y may be deduced from (A 7), (A 8) and is as follows:

(i) when $y^2 = O(1-x)^{\frac{5}{3}}$, then $\theta \approx 0$, and H is a function of the similarity variable $\eta = y(1-x)^{-\frac{5}{6}}$ alone: indeed it is exactly the same function as would have been found by conventional similarity arguments;

(ii) when $(1-x)^{\frac{4}{3}} \gg y^2 \gg (1-x)^{\frac{5}{3}}$ then $\theta \ll 1$, and

$$H \approx \frac{560}{27}\left(\frac{3(1-x)^{\frac{5}{6}}}{y}\right)^8, \tag{A 9}$$

so that H has an algebraic decay with respect to η;

(iii) when $y^2 \sim (1-x)^{\frac{4}{3}}$ the similarity solution is not correct and H has the form

$$\frac{1}{\sqrt{\pi}}\left[\frac{12(1-x)^{\frac{5}{3}}}{y^2}\right]^4 \int_{\frac{1}{12}y^2(1-x)^{-\frac{4}{3}}}^\infty e^{-\mu}\mu^{\frac{7}{2}}\,d\mu; \tag{A 10}$$

(iv) finally, when $y^2 \gg (1-x)^{\frac{4}{3}}$ then

$$H \approx \frac{2\ 3^{\frac{1}{2}}}{\sqrt{\pi}}\frac{(1-x)^2}{y}\exp\left[-\frac{y^2}{12(1-x)^{\frac{4}{3}}}\right], \tag{A 11}$$

and decays exponentially.

If B is non-zero a solution of (A 5) subject to the boundary conditions (A 6) is not so simple, but may be obtained formally by the Laplace-transform method. This formal solution also leads to the conclusions (i)–(iv).

REFERENCES

Ackerberg, R. C. 1965 *J. Fluid Mech.* **21**, 47.

Falkner, V. M. & Skan, S. W. 1930 *Aero. Res. Comm. R & M* no. 1314.

Goldstein, S. 1939 *Proc. Camb. Phil. Soc.* **35**, 338.

Goldstein, S. 1965 *J. Fluid Mech.* **21**, 33.

Görtler, H. 1955 Tables of universal functions of the new series. Math. Inst. Univ. Freiburg.

Görtler, H. 1957 *J. Math. Mech.* **6**, 1.

Hartree, D. R. 1937 *Proc. Camb. Phil. Soc.* **33**, 223.

Rosenhead, L. 1963 *Laminar Boundary Layers*. Oxford University Press.

Schlichting, H. 1960 *Boundary Layer Theory*. New York: McGraw-Hill.

Stewartson, K. 1954 *Proc. Camb. Phil. Soc.* **50**, 454.